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Ternary codes from the strongly regular (45, 12, 3, 3) graphs and orbit matrices of 2-(45, 12, 3) designs

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ABSTRACT

The enumeration of strongly regular graphs with parameters (45, 12, 3, 3) has been completed, and it is known that there are 78 non-isomorphic strongly regular (45, 12, 3, 3) graphs. A strongly regular graph with these parameters is a symmetric (45, 12, 3) design having a polarity with no absolute points. In this paper we examine the ternary codes obtained from the adjacency (resp. incidence) matrices of these graphs (resp. designs), and those of their corresponding derived and residual designs. Further, we give a generalization of a result of Harada and Tonchev on the construction of non-binary self-orthogonal codes from orbit matrices of block designs under an action of a fixed-point-free automorphism of prime order. Using the generalized result we present a complete classification of self-orthogonal ternary codes of lengths 12, 13, 14, and 15, obtained from non-fixed parts of orbit matrices of symmetric (45, 12, 3) designs admitting an automorphism of order 3. Several of the codes obtained are optimal or near optimal for the given length and dimension. We show in addition that the dual codes of the strongly regular (45, 12, 3, 3) graphs admit majority logic decoding.

1. Introduction

The quest to enumerate the strongly regular (45, 12, 3, 3) graphs began in [22] where Mathon and Spence found 58 pairwise non-isomorphic such graphs. Much later Coolsaet, De Jager and Spence, in [10] through an exhaustive computer search established the existence of 78 non-isomorphic strongly regular graphs with these parameters, thus settling the enumeration problem.

A strongly regular (45, 12, 3, 3) graph is a regular graph on 45 vertices of degree 12 such that each pair of distinct vertices has 3 common neighbours. Since $\lambda = \mu = 3$, we may associate with every strongly regular (45, 12, 3, 3) graph a (45, 12, 3) design in such a way that the 45 vertices serve as both points and blocks of the design and adjacency in the graph is incidence in the design, and thus throughout the paper we will indistinguishably refer to either expressions to mean the same structure. Interchanging points and blocks that correspond to the same vertex gives rise to a polarity in the design for which no point is absolute. Conversely, every 2-(45, 12, 3) design with a polarity without absolute points corresponds to a strongly regular (45, 12, 3, 3) graph. Symmetric (45, 12, 3) designs belong to the series with parameters

$$v = q^{l+1}\left(1 + \frac{q^{l+1}-1}{q-1}\right), \qquad k = q^l\,\frac{q^{l+1}-1}{q-1} \quad \text{and} \quad \lambda = q^l\,\frac{q^l-1}{q-1}, \tag{1}$$

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where q is any prime power and l is any positive integer (see [2], p. 622 with $q = 3$ and $l = 1$). Altogether, the total number of nonisomorphic (45, 12, 3) designs found in [22] is 3752. Many of these designs are self-dual and each of these self-dual designs possesses a polarity. Some have polarities with no absolute points, giving rise to strongly regular (45, 12, 3, 3) graphs.

The purpose of this paper is two-fold: First, we examine the ternary codes obtained from the adjacency (resp. incidence) matrices of the strongly regular (45, 12, 3, 3) graphs (resp. designs), and those of their corresponding derived and residual designs. This study is carried out with the aim to provide a distinguishing property that characterizes the codes of this class of graphs (resp. designs). In addition, our study emerges from the knowledge that codes obtained from the row span of adjacency matrices of strongly regular graphs admit an efficient decoding algorithm, known as majority logic decoding [27]. Second, an active area of research in coding theory is the classification of self-orthogonal codes of given length or dimension. For small parameters, self-orthogonal codes are generally classified under an action of an automorphism on the design from which these originate; see [16] or [4]. Several methods and techniques, among which the method of orbit matrices have been used for this purpose. Tonchev in [28, Theorem 1.113] (see Result 2 in this paper) introduced a method of constructing non-binary self-orthogonal codes from orbit matrices under an action of an automorphism of prime order. Using the non-fixed part of an orbit matrix to construct non-binary self-orthogonal codes, this paper proposes a generalization of [28, Theorem 1.113] which is applicable to symmetric block designs. As an illustration of this generalizing method we present a complete classification of self-orthogonal ternary codes of lengths 12, 13, 14, and 15, obtained from orbit matrices of symmetric (45, 12, 3) designs admitting an automorphism of order 3. In several cases, we obtain codes with parameters which are as good as the parameters of the presently best-known codes (some of which are optimal) or near optimal according to [6,13]; see also [14].

The paper is organized as follows: after a brief description of our terminology and some background, in Section 3 we present our results on the ternary codes associated with the adjacency matrices of the strongly regular (45, 12, 3, 3) graphs (Proposition 1), and a distinguishing property of the codes from this class of graphs (Proposition 3). In Section 4 we study ternary codes generated by the incidence matrices of derived and residual designs of the 78 symmetric (45, 12, 3) designs, and in Section 5 we give a brief but complete overview of orbit matrices with aims to be used in Section 6 where we present a generalization result that deals with the construction of non-binary self-orthogonal codes from orbit matrices of symmetric (v, k, λ) designs admitting an automorphism of prime order (Theorem 4). As an illustration of Theorem 4, in Sections 7–10 we determine and completely classify all the ternary self-orthogonal codes of lengths 12, 13, 14 and 15, obtained from the orbit matrices of the symmetric (45, 12, 3) designs examined in the paper.

2. Background and terminology

We assume that the reader is familiar with some basic notions and elementary facts from strongly regular graphs, design and coding theory. For an account of these topics and an interplay we refer the reader to [1] or [2]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a $t - (v, k, \lambda)$ design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. If there exist two blocks B_1 and B_2 such that $\{P \in \mathcal{P} \mid P\mathcal{I}B_1\} = \{P \in \mathcal{P} \mid P\mathcal{I}B_2\}$, then we say that the design $\mathcal{D}$ contain repeated blocks. A design that has no repeated blocks is called a simple design. If $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a simple design, we can identify a block B with the subset $\{P \in \mathcal{P} \mid P\mathcal{I}B\}$ of $\mathcal{P}$. The design is *symmetric* if it has the same number of points and blocks. In a 2-(v, k, λ) design every point is incident with exactly $r = \frac{\lambda(v-1)}{k-1}$ blocks, and r is called the replication number of a design. The number $n = r - \lambda$ is called the order of a 2-(v, k, λ) design. If a block design is symmetric, then $r = k$. Given a symmetric 2-(v, k, λ) design $\mathcal{D}$, a *residual* design of $\mathcal{D}$ is the design obtained by deleting a block of $\mathcal{D}$ and retaining those points not incident with the block. A residual design at any block of $\mathcal{D}$ is a 2-$(v - k, k - \lambda, \lambda)$ design. A *derived* design of $\mathcal{D}$ with respect to a block is the design obtained by deleting a block and retaining those points incident with the block. A derived design of $\mathcal{D}$ with respect to a block is a 2-$(k, \lambda, \lambda - 1)$ design. An automorphism of a design $\mathcal{D}$ is a permutation on $\mathcal{P}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by Aut $\mathcal{D}$.

The *code C_F of the design $\mathcal{D}$* over the finite field F is the space spanned by the incidence vectors of the blocks over F. If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by $\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F. All our codes will be *linear codes*, i.e. subspaces of the ambient vector space. If a code C over a field of order q is of length n, dimension k, and minimum weight d, then we write $[n, k, d]_q$ to show this information. The *support*, $\mathrm{Supp}(v)$, of a vector v is the set of coordinate positions where the entry in v is non-zero. So $|\mathrm{Supp}(v)| = \mathrm{wt}(v)$, where $\mathrm{wt}(v)$ is the weight of v. An $[n, k]$ linear code C is said to be a *best known linear* $[n, k]$ *code* if C has the highest minimum weight among all known $[n, k]$ linear codes. An $[n, k]$ linear code C is said to be an *optimal linear* $[n, k]$ code if the minimum weight of C achieves the theoretical upper bound on the minimum weight of $[n, k]$ linear codes, and *near-optimal* if its minimum distance is at most 1 less than the largest possible value. The weight enumerator of C is defined as $W_C(x) = \sum_{i=0}^{n} A_i x^i$, where A_i denotes the number of codewords of weight i in C. The *dual* code $C^{\perp}$ is the orthogonal complement under the standard inner product $(\cdot, \cdot)$, i.e. $C^{\perp} = \{v \in F^n | (v, c) = 0 \text{ for all } c \in C\}$. A code C is *self-orthogonal* if $C \subseteq C^{\perp}$. The all-one vector will be denoted by $\mathbf{1}$, and is the constant vector of weight the length of the code, and whose coordinate entries consist entirely of 1's. An *automorphism* of a code is any permutation of the coordinate positions that maps codewords to codewords. Two codes are *equivalent* if one of the codes can be obtained from the other by permuting the coordinates and permuting the symbols within one or more coordinate positions. An *automorphism* of a code is any permutation of the coordinate positions that maps codewords to codewords and will be denoted by Aut (C).

For the graphs our terminology is standard: the graphs, $\mathcal{G} = (V, E)$ with vertex set V and edge set E, are undirected and the *degree* of a vertex is the number of edges containing the vertex. A graph is *regular* if all the vertices have the same degree; a regular graph is *strongly regular* of type (n, k, λ, μ) if it has n vertices, degree k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices. The complementary graph of a strongly regular graph with parameters (n, k, λ, μ) is a strongly regular graph with parameters $(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$. The *neighbourhood design* of a regular graph is the 1-design formed by taking the points to be the vertices of the graph and the blocks to be the sets of neighbours of a vertex, for each vertex.

The *code* of a graph Γ over a finite field F is the row span of an adjacency matrix A over the field F, denoted by $C_F(\Gamma)$ or $C_F(A)$, it is also the code of the neighbourhood design. The *dimension* of the code is the rank of the matrix over F, also written $\text{rank}_p(A)$ if $F = \mathbb{F}_p$, in which case we will speak of the *p-rank* of A or Γ, and write $C_p(\Gamma)$ or $C_p(A)$ for the code. A connected strongly regular graph has diameter 2. If v and w are vertices of a connected strongly regular graph Γ such that $d(v, w) = i$, $i = 0, 1, 2$, then the number p_{ij} of neighbours of w whose distance from v is j, $j = 0, 1, 2$, are the intersection numbers of Γ. The 3×3-matrix with entries p_{ij}, $i, j = 0, 1, 2$, is called the *intersection matrix* of Γ.

3. Ternary codes from the (45, 12, 3) designs

Let Γ_i where $1 \le i \le 78$ denote any of the 78 strongly regular (45, 12, 3, 3) graphs examined in the paper. Then the adjacency matrix of Γ_i is also the incidence matrix of a symmetric (45, 12, 3) design (with a polarity having no absolute points). We will use both views interchangeably throughout the paper. Note that given a design and any prime p, the p-ary code of the design is the code over $\mathbb{F}_p$ generated by the rows of the incidence matrix (which are the characteristic functions of the blocks). If A is an incidence matrix of a 2-(v, k, λ) design and $\text{rank}_p(A) < v - 1$, then it is well known that this code is interesting only when p divides $r - \lambda$, the order of the design (see [28, Theorem 1.86]). Since the order of each design is 9, only the ternary codes of such designs will be of interest for characterization. Thus, taking the ternary row span of the adjacency (resp. incidence) matrix of Γ_i we use Magma [8,3] to construct the codes which we examine in the sequel, and denote these C_{Γ_i}. However, we may omit the subscript i when the context is clear.

Tables 1 and 2 summarize some invariants of the codes C_{Γ_i}. In these tables the 3-rank is the rank of the adjacency matrix of Γ_i over $\mathbb{F}_3$, $|\text{Aut}(\Gamma_i)|$, $|\text{Aut}(C_{\Gamma_i})|$ are the orders of the automorphism groups for each design and code respectively; and the remaining is the code's weight distribution.

The ordering of the codes follows that for the strongly regular (45, 12, 3, 3) graphs as given by [26]. Note that there is only one point-transitive group. This group is isomorphic to $U_4(2) : 2$ and is the group of design Γ_5. It has been proved in [24, Theorem 3.3] and also in [12] that Γ_5 is the unique point-primitive, flag-transitive symmetric (45, 12, 3) design; see also [5,11,25] for related results on this design. The second column gives the 3-rank. The attentive reader would notice the minimality of the 3-rank of the design Γ_5; the 3-rank 15 of Γ_5 is minimal among the other designs with the same parameters; a property that is enjoyed by the codes of the geometric design. This follows by a famous conjecture of Hamada which reads as follows:

Conjecture 1 (*Hamada's Conjecture*)**.** *Let $\mathcal{D}$ be a design with the parameters of a geometric design $PG_d(n, q)$ or $AG_d(n, q)$, where q is a power of a prime p. Then the p-rank of the incidence matrix of $\mathcal{D}$ is greater than or equal to the p-rank of the corresponding geometric design. Moreover, equality holds if and only if $\mathcal{D}$ is isomorphic to the geometric design.*

For more information about Hamada's conjecture we refer the reader to [15]. We also observe that the code of the design with largest automorphism group is not the code of design Γ_5 whose order is 51 840, but the code of design Γ_{15} with automorphism group of order 271 177 280. We found that the codes with the same weight distribution were in all instances equivalent. For each equivalence class in Tables 1 and 2 we present a representative of the class of mutually equivalent codes. The sets of mutually equivalent codes that contain more than one code are {2, 6}, {5, 9}, {7, 8}, {25, 67, 70, 76}, {27, 57}, {28, 59}, {31, 34, 50}, {32, 36, 44}, {35, 41, 42, 43}, {38, 51}, {40, 52}, {55, 66}, {58, 65} and {60, 69}.

Proposition 1. *Let C_Γ denote a code obtained from the ternary row span of an adjacency matrix of Γ, where Γ is any of the* 78 *strongly regular graphs with parameters* (45, 12, 3, 3)*. Then the following holds:*

(a) *C_Γ is a self-orthogonal code. Moreover, $\mathbf{1} \in C_\Gamma$ and $\mathbf{1} \in {C_\Gamma}^\perp$;*
(b) *the minimum weight of ${C_\Gamma}^\perp$ is* 6*;*
(c) *the minimum weight of C_Γ is* 6*, except when $\Gamma = \Gamma_5$ or $\Gamma = \Gamma_{24}$ in which case the minimum weight is* 12*.*

Proof. We will use the parameters of Γ to show (a), i.e., that C_Γ is self-orthogonal. Notice first that Γ is a strongly regular graph with parameters (45, 12, 3, 3) and intersection matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 12 & 3 & 3 \\ 0 & 8 & 9 \end{bmatrix}.$$

Table 1
Weight distributions of the ternary codes from srg (45, 12, 3, 3) graphs.

Γ_i	3-rank	$\lvert\mathrm{Aut}(\Gamma_i)\rvert$	$\lvert\mathrm{Aut}(C_{\Gamma_i})\rvert$	0	6	9	12	15	18	21	24
1	19	12	12	1	8	12	774	19 288	547 604	9 183 132	74 914 716
2	17	216	216	1	4		234	3 872	65 312	977 616	8 374 668
3	19	12	12	1	8	12	690	18 568	545 564	9 200 448	74 902 512
4	17	432	432	1	4		222	3 032	62 384	997 740	8 362 608
5	15	51 840	51 840	1			90	1 152	8 660	92 340	952 020
7	19	48	288	1	16		1 242	21 536	556 676	9 111 204	74 980 068
10	17	12	12	1	2	26	402	2 608	57 456	102 104	8 379 012
11	17	108	324	1	12	22	342	2 904	55 468	1 018 116	8 372 772
12	17	108	1 944	1	12	10	738	2 040	63 112	979 344	8 451 504
13	19	1	972	1	36	94	3 582	26 064	504 964	9 174 492	75 156 012
14	19	2	23 328	1	54	94	3 474	25 344	511 552	9 146 628	75 232 908
15	19	20	226 748 160	1	120	10	6 570	23 640	530 320	8 976 960	75 601 800
16	19	12	52 488	1	66	154	3 330	25 368	510 088	9 146 088	75 266 460
17	19	4	36	1	18	70	2 286	24 696	520 108	9 175 140	75 063 672
18	19	8	72	1	32	74	2 190	24 616	521 616	9 165 384	75 101 520
19	19	2	12	1	20	86	2 262	24 304	522 504	9 166 140	75 088 272
20	19	4	4	1	8	12	630	19 216	551 564	9 171 252	74 923 356
21	19	4	4	1	8	12	618	18 376	548 636	9 191 376	74 911 296
22	17	8	8	1	4		174	3 776	65 384	979 884	8 371 644
23	19	8	8	1	4	66	2 382	24 776	518 600	9 184 896	75 025 824
24	17	192	192	1			138	2 496	65 060	997 092	8 349 948
25	20	3	12	1	24	40	1 674	51 960	1 629 604	27 687 744	224 585 604
26	20	18	108	1	18	76	1 656	52 452	1 625 014	27 702 216	224 563 752
27	20	2	12	1	24	46	1 494	52 896	1 640 380	27 640 008	224 613 738
28	20	6	72	1	24	46	1 818	53 256	1 630 300	27 668 844	224 595 810
29	20	1	31 104	1	60	178	6 858	73 848	1 562 848	27 525 744	225 154 278
30	18	3	108	1	18	58	1 134	8 136	172 072	3 039 768	25 114 482
31	20	1	45 349 632	1	138	262	10 566	78 432	1 515 352	27 522 396	225 467 946
32	20	2	11 664	1	72	238	6 714	73 872	1 561 384	27 525 204	225 187 830
33	18	54	1 944	1	30	46	1 062	8 448	171 184	3 039 012	25 127 730
35	20	1	162	1	42	178	6 966	74 568	1 556 260	27 553 608	225 077 382
37	20	6	12	1	18	76	1 440	51 876	1 634 230	27 675 000	224 590 104
38	20	2	1 944	1	54	130	6 930	75 024	1 555 660	27 552 744	225 080 478
39	18	6	36	1	12	58	1 170	8 736	169 876	3 049 056	25 088 850
40	18	6	324	1	24	58	1 098	8 544	168 436	3 053 808	25 085 682
45	19	2	2	1	8	4	446	18 728	547 396	9 200 244	74 885 436
46	19	1	1	1	8	10	482	18 884	546 664	9 200 028	74 888 370
47	19	2	12	1	16	12	636	21 776	556 436	9 120 798	74 981 778
48	19	2	2	1	4	8	432	17 596	544 804	9 224 370	74 858 454
49	19	2	12	1	14	32	612	22 012	554 694	9 124 272	74 983 890
53	19	6	6	1	8	12	636	17 200	541 856	9 233 334	74 863 038
54	20	18	36	1	18	64	1 620	52 020	1 629 742	27 688 068	224 579 952
55	20	2	12	1	24	40	1 458	51 528	1 637 380	27 667 008	224 594 676
56	20	6	12	1	18	64	1 404	51 660	1 637 014	27 668 628	224 588 160
58	20	18	23 328	1	48	10	3 114	60 864	1 654 492	27 459 216	224 783 442
60	18	162	324	1	12	10	414	9 600	189 040	2 993 328	25 046 622
61	18	27	209 952	1	48	10	2 142	7 080	183 208	2 964 816	25 277 526
62	19	6	36	1	16		1 080	20 672	552 356	9 143 766	74 940 378
63	20	3	36	1	24	46	1 656	53 364	1 633 756	27 656 694	224 609 094
64	19	6	6	1	8	12	648	18 076	544 784	9 211 914	74 881 146
68	20	9	54	1	18	58	1 602	51 804	1 632 106	27 680 994	224 588 052
71	19	6	6	1	4	6	438	17 492	545 348	9 223 722	74 856 138
72	17	54	54	1	4		180	3 584	63 872	988 470	8 361 438
73	19	2	2	1	8	2	434	18 156	546 824	9 210 018	74 871 402
74	19	2	2	1	8	12	540	18 028	547 532	9 204 570	74 882 370
75	18	54	216	1	12	4	378	8 232	186 040	3 020 328	25 027 560
77	19	3	3	1	8	12	594	18 112	545 936	9 207 864	74 885 574
78	20	3	12	1	24	64	1 422	53 148	1 639 696	27 639 252	224 623 674

Now, if we fix a vertex v in Γ we can divide the remaining vertices into two sets, namely Γ' of size 12 and Γ'' of size 32, with Γ' being the set of vertices adjacent to v, and Γ'' the set of vertices non-adjacent to v. Now, from the second column of the above matrix we deduce that each vertex in Γ' is adjacent to v and to 3 other vertices in Γ', thus to 8 vertices in Γ''; and the third column shows that a vertex in Γ'' is adjacent to 3 vertices in Γ', and so to 9 vertices in Γ''. The valency 12 ensures that generating codewords have length zero (mod 3) and the two 3's ensure that (i) any two generating codewords have 3 non-zero entries in common, and (ii) that any two generating codewords are orthogonal to one another. Hence C_Γ is self-orthogonal. Also **1** is orthogonal to the codewords corresponding to the blocks of Γ (viewed as a design), since these codewords have weights divisible by 3, thus showing $\mathbf{1} \in C_\Gamma^{\perp}$. That $\mathbf{1} \in C_\Gamma$ follows since the sum (modulo 3) of all rows of a

Table 2
Weight distributions of the ternary codes from srg (45, 12, 3, 3) graphs (continued).

i	27	30	33	36	39	42	45
1	272 098 064	436 155 624	292 373 806	71 603 190	5 273 352	91 172	724
2	30 317 024	48 211 740	32 712 698	7 877 370	585 072	14 152	400
3	272 033 768	436 305 276	292 240 966	71 653 014	5 270 736	89 192	712
4	30 239 624	48 388 608	32 552 066	7 943 106	577 632	12 724	412
5	3 394 640	5 270 400	3 712 770	850 170	63 360	3 060	244
7	272 306 576	435 633 552	292 834 754	71 441 850	5 272 992	100 132	868
10	30 113 096	48 625 506	32 325 004	8 064 394	543 432	16 772	348
11	30 107 528	48 647 196	32 300 592	8 077 622	540 576	16 500	512
12	30 035 744	48 676 896	32 280 828	8 109 854	517 176	22 548	356
13	271 598 552	436 836 132	291 585 816	72 258 062	4 967 064	149 436	1160
14	271 457 288	437 007 366	291 451 608	72 322 538	4 950 648	150 588	1376
15	271 177 280	437 057 964	291 330 060	72 609 830	4 747 320	197 940	1652
16	271 345 976	437 182 866	291 295 932	72 399 626	4 933 296	150 204	2012
17	271 815 056	436 517 478	291 949 632	71 971 202	5 095 368	125 748	992
18	271 730 048	436 632 372	291 855 004	72 016 474	5 085 096	125 732	1308
19	271 768 784	436 575 240	291 902 896	71 994 610	5 088 912	126 404	1032
20	272 119 232	436 098 168	292 436 446	71 566 470	5 284 512	89 948	652
21	272 041 832	436 275 036	292 275 814	71 632 206	5 277 072	88 520	664
22	30 310 688	48 229 092	32 698 946	7 879 782	586 896	13 528	364
23	271 900 064	436 402 584	292 044 260	71 925 930	5 105 640	125 764	676
24	30 271 088	48 343 824	32 598 450	7 910 058	591 072	10 644	292
25	815 988 428	1 309 337 622	876 337 446	215 079 692	15 824 916	258 198	1448
26	816 003 764	1 309 335 948	876 334 986	215 077 226	15 829 632	255 780	1880
27	816 143 336	1 308 975 336	876 684 180	214 916 858	15 857 496	257 100	1508
28	816 082 856	1 309 129 560	876 516 780	215 016 542	15 825 744	261 276	1544
29	815 560 928	1 309 377 312	875 997 912	215 844 266	15 299 064	378 924	2180
30	90 400 904	145 785 258	97 054 128	24 154 034	1 639 440	50 292	764
31	814 797 656	1 310 506 722	874 756 788	216 776 186	14 899 752	448 788	3416
32	815 449 616	1 309 552 812	875 842 236	215 921 354	15 281 712	378 540	2816
33	90 362 168	145 842 390	97 006 236	24 175 898	1 635 624	49 620	1040
35	815 702 192	1 309 206 078	876 132 120	215 779 790	15 315 480	377 772	1964
37	816 027 956	1 309 245 228	876 439 530	215 014 802	15 848 640	253 764	1736
38	815 699 744	1 309 204 026	876 138 120	215 774 042	15 318 432	376 956	2060
39	90 447 992	145 728 180	97 098 864	24 132 542	1 644 912	49 908	692
40	90 435 464	145 760 688	97 063 824	24 152 198	1 639 800	50 028	836
45	272 082 896	436 228 440	292 320 214	71 599 838	5 291 320	85 828	668
46	272 079 332	436 231 842	292 313 716	71 608 004	5 286 640	86 806	680
47	272 225 252	435 788 550	292 722 074	71 460 666	5 288 340	94 222	910
48	272 030 708	436 377 438	292 183 406	71 650 306	5 289 768	83 570	602
49	272 206 004	435 822 984	292 690 906	71 475 634	5 285 448	93 884	1080
53	271 973 612	436 490 442	292 069 354	71 715 438	5 269 980	85 862	694
54	816 007 796	1 309 303 116	876 374 838	215 054 054	15 835 824	255 564	1724
55	816 042 860	1 309 210 614	876 472 230	214 999 988	15 850 404	254 742	1448
56	816 059 204	1 309 185 180	876 497 526	214 983 854	15 856 776	253 332	1580
58	816 717 464	1 307 554 272	877 908 048	214 526 366	15 827 688	287 292	2084
60	90 857 024	144 945 720	97 856 160	23 724 710	1 760 688	36 372	788
61	90 250 496	145 857 780	96 976 032	24 267 086	1 566 504	66 972	788
62	272 252 468	435 805 110	292 677 614	71 495 958	5 274 612	96 622	814
63	816 091 928	1 309 088 736	876 565 704	214 986 572	15 835 140	260 196	1490
64	272 037 404	436 331 718	292 214 866	71 657 478	5 275 152	87 578	682
68	816 009 812	1 309 286 700	876 394 764	215 042 468	15 838 920	255 456	1646
71	272 040 284	436 361 382	292 198 772	71 641 464	5 292 696	83 098	622
72	30 298 988	48 268 926	32 660 318	7 895 406	585 612	12 982	382
73	272 069 900	436 277 130	292 275 856	71 613 244	5 293 456	84 366	670
74	272 063 612	436 275 774	292 272 178	71 624 682	5 285 088	86 438	634
75	90 756 548	145 180 998	97 644 210	23 807 840	1 753 596	34 014	728
77	272 040 428	436 318 110	292 231 174	71 647 488	5 278 284	87 218	664
78	816 116 120	1 309 014 000	876 652 374	214 931 474	15 854 652	256 848	1652

generator matrix G of C is the all-one vector. Now, from [10, Section 3] we have that the eigenvalues of an adjacency matrix A of Γ are $\theta_0 = 12$, $\theta_1 = 3$, and $\theta_2 = -3$ with corresponding multiplicities $f_0 = 1, f_1 = 20$ and $f_2 = 24$. Since $p \mid (\theta_1 - \theta_2)$ i.e., $3 \mid 6$, we have from [7, Section 3] that $\mathrm{rank}_3(\Gamma) \leq \min(f_1 + 1, f_2 + 1) = 21$, and computations with Magma give us the respective 3-rank in each case as listed in Table 1.

(b) Denote by $d^{\perp}$ the minimum weight of $C^{\perp}$. From [1, Lemma 2.4.2] we have that $d^{\perp} \geq \frac{r}{\lambda} + 1 = \frac{12}{3} + 1 = 5$. If $d^{\perp} = 5$, we argue as follows to get a contradiction. Let p be a fixed point in the support S of a non-zero codeword $u \in {C_{\Gamma}}^{\perp}$ of weight $s = d^{\perp}$ and p_i be the number of blocks of the design Γ passing through p and meeting S in i points. A counting argument gives

$$\sum_{i=1}^{k} p_i = r, \quad \sum_{i=2}^{k}(i-1)p_i = (s-1)\lambda. \tag{2}$$

From Eq. (2) we obtain

$$\sum_{i=3}^{k}(i-2)p_i = (s-1)\lambda - r, \tag{3}$$

and Eqs. (2) and (3) imply that $p_2 = r - \sum_{i=3}^{k} p_i \geq r - \sum_{i=3}^{k}(i-2)p_i = r - [(s-1)\lambda - r] = 2r - (s-1)\lambda$. Hence we have $p_2 \geq 24 - 12 = 12$ for any point of S. As in [19] we now examine the entries of u. Consider $S = \{q_i \mid 1 \leq i \leq 5\}$, since $\mathbf{1} \in {C_\Gamma}^\perp$ we must have entries $+1$ at four points, say q_i, for $i = 1, \ldots, 4$ or $i = 1, 2, 3, 4$; and -1 at q_5. However, every block meeting S in two points and passing through q_1 must pass through q_5, but there are only four points remaining once q_1 is chosen; thus not all twelve blocks which meet S in two points can pass through q_5; thus we have a contradiction, and so $d^\perp \geq 6$. Now, direct calculations show that the weights of the rows of the generator matrix for ${C_\Gamma}^\perp$ equal 6, so that the minimum weight $d^\perp \leq 6$, and the assertion follows.

(c) Follows from part (a) since $C_\Gamma \subseteq {C_\Gamma}^\perp$. For the two exceptional cases, i.e., $\Gamma = \Gamma_5$ or $\Gamma = \Gamma_{24}$, we used Magma to ascertain that the minimum weight is as stated. Moreover, for $\Gamma = \Gamma_5$ the minimum weight of C_Γ is the block size of Γ and the words of minimum weight are scalar multiples of the incidence vectors of the blocks. □

In the following we make some observations related to the codes, in particular to $C_{\Gamma_5} = [45, 15, 12]_3$.

Remark 1. (i) Since $\mathbf{1} \in C_\Gamma$ then the code of the complementary design 2-(45, 33, 24) is C_Γ.

(ii) Unless Γ is the point-primitive design, the supports of constant words of weight 12 in C_{Γ_5} are not blocks of a design. Recall that one of the designs, i.e., the design Γ_5 possesses a transitive automorphism group, namely $U_4(2) \cdot 2$, thus C_{Γ_5} is invariant under a transitive automorphism group. We deduce from this that the support of the codewords of weight 12 in C_{Γ_5} hold a self-dual symmetric 1-(45, 12, 12) design. This is in fact the unique point-primitive flag transitive symmetric 2-(45, 12, 3) design (see Eq. (1)) whose point set $\mathcal{P}$ consists of all anisotropic 1-dimensional subspaces $U = \langle u \rangle \leq V$ satisfying $f(u, u) = 1$ in a non-degenerate orthogonal space (V, f) of dimension $2l + 1$ over $\mathbb{F}_3$ with discriminant $(-1)^l$, and blocks having the form $B(U) = \{W \in \mathcal{P} \mid f(U, W) = 0\}$, $U \in \mathcal{P}$; see [12, Theorem 3.13], and also [5, p. 7, item (4^0)] and [24, Theorem 3.3]. Notice that this identification occurs since $U_4(2) \cong P\Omega(5, 3)$. The codewords of weight 12 in C_{Γ_5} split into two orbits of equal length 45, that are stabilized by maximal subgroups of $U_4(2) \cdot 2$ of type $(2^{\cdot}(A_4 \times A_4) \cdot 2) \cdot 2$. The code C_{Γ_5} is far from optimal, but its dual code ${C_{\Gamma_5}}^\perp = [45, 30, 6]_3$ is at distance 1 less than the optimal, thus a near-optimal code.

(iii) The 1152 codewords of weight 15 in C_{Γ_5} split into 4 orbits of lengths 36, 36, 540, and 540 respectively under the action of $U_4(2) \cdot 2$. The support of a codeword of weight 15 in an orbit of length 36, produces a 1-(45, 15, 12) design with 36 blocks whose ternary row span gives rise to an irreducible self-orthogonal $[45, 14, 15]_3$ code $\mathcal{E}$ of dimension 14 which is in fact a code of codimension 1 in C_{Γ_5}. The dual $\mathcal{E}^\perp$ of $\mathcal{E}$ is a $[45, 31, 5]_3$ code. The stabilizer of a codeword of weight 15 in either of the above orbits is a maximal subgroup of type $S_6 \times 2$ in $U_4(2) \cdot 2$ of order 1440. Furthermore, the support of a codeword of weight 15 in an orbit of length 540 produces a 1-(45, 15, 180) with 540 blocks, whose code is a $[45, 44, 2]_3$. The irreducibility of $\mathcal{E}$ follows since the 3-modular character table of the group $U_4(2)$ is completely known (see [18]). It also follows from this that the irreducible 14-dimensional $\mathbb{F}_3$-representation is unique. Since $\mathrm{Aut}(\mathcal{E})$ contains $U_4(2)$, by using the weight enumerator given below we can easily see that $\mathcal{E}$, under the action of $U_4(2)$, does not contain an invariant subspace of dimension 1. So if $\mathcal{E}$ is reducible, it must contain an invariant subspace U of dimension m where $2 \leq m \leq 13$. However, calculations with Meat-Axe within Magma [8,3] show that the module $14 \cdot (1 \oplus 1 \oplus 1)$ (with no trivial submodules) occurs naturally as a submodule of $U_4(2)$ acting on the cosets of $2^{\cdot}(A_4 \times A_4) \cdot 2$. Hence $\mathcal{E}$ is the 14-dimensional $\mathbb{F}_3$-module on which $U_4(2)$ acts irreducibly. Using Magma we found the weight enumerator of $\mathcal{E}$ which follows

$$\begin{aligned} W_{\mathcal{E}}(x) = {} & 1 + 72x^{15} + 6420x^{18} + 19\,440x^{21} + 336\,060x^{24} + 1\,109\,420x^{27} + 1\,781\,136x^{30} + 1\,215\,720x^{33} \\ & + 295\,170x^{36} + 18\,360x^{39} + 1080x^{36} + 90x^{45}. \end{aligned}$$

(iv) Finally the codewords of weight 18 in C_{Γ_5} split into 9 orbits of lengths 40, 40, 480, 540, 720, 1080, 1080, 1440, and 3240, respectively. The support of a codeword of weight 18 in the unique orbit of length 480 gives rise to a 1-(45, 18, 96) design $\mathcal{L}$ with 240 blocks whose code is an irreducible 25-dimensional $[45, 25, 6]_3$ code. The codewords of weight 18 are in fact the incidence vectors of the blocks of the design. This code meets its dual $[45, 20, 8]_3$ in the self-orthogonal subcode $[45, 15, 12]_3$ isomorphic to C_{Γ_5}. An argument similar to the one used earlier can be given to show the irreducibility of this latter code, so we leave it to the reader.

The rows of the adjacency matrix of Γ can be used as orthogonal parity checks that allow majority decoding of ${C_\Gamma}^\perp$ up to its full error-correcting capacity. Using this we prove the following.

Table 3
Enumeration of the derived 2-(12, 3, 2) designs.

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	No	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	No	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	No
576	1	48	1	8	5
432	1	36	2	6	10
144	1	18	3	4	8
72	2	16	5	3	1
64	1	12	1	2	66
54	1	9	1	1	165

Table 4
Weight distributions of the ternary codes of the derived designs.

No.	Parameters	$\lvert\mathrm{Aut}(C_\Gamma)\rvert$	0	2	3	4	5	6	7	8	9	10	11	12
1	$[12, 11, 2]_3$	479 001 600	1	132	440	2970	7920	20 328	33 264	42 570	37 400	22 572	8184	1366
2	$[12, 10, 2]_3$	82 944	1	36	152	1026	2592	6 720	11 232	14 148	12 344	7 668	2664	466

Table 5
Enumeration of the residual 2-(33, 9, 3) designs.

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	No	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	No	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	No	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	No
1152	1	36	3	12	4	4	27
192	1	24	1	9	2	3	102
72	3	18	24	8	6	2	233
48	2	16	1	6	59	1	847

Proposition 2. *$C_\Gamma{}^\perp$ can correct up to 2 errors by majority decoding.*

Proof. It follows from [27, Theorem 2.1] since for Γ we have $\frac{k+\max(\lambda,\mu)-1}{2\cdot\max(\lambda,\mu)} = \left\lfloor \frac{12+3-1}{2\cdot 3} \right\rfloor = 2$. $\square$

We have remarked earlier that the codes with the same weight distribution were in all instances equivalent, so using this we deduce the following.

Proposition 3. *The ternary codes of the* 78 *non-isomorphic strongly regular* $(45, 12, 3, 3)$ *graphs can be distinguished by their automorphism groups or by the weight distribution or by their multisets. Up to equivalence there are* 58 *non-isomorphic ternary self-orthogonal codes of length* 45 *obtained from these graphs.*

4. Ternary codes of derived and residual designs

Using [20,23] we have determined that for the 78 symmetric 2-(45, 12, 3) designs there are 275 mutually non-isomorphic derived designs with parameters 2-(12, 3, 2). Incidence matrices of these designs span up to equivalence 2 ternary codes of length 12. These codes are optimal; see [14]. In Table 3 we give an information about orders of the automorphism groups of the 275 derived designs, and in Table 4 we give the weight distributions and orders of the automorphism groups of the corresponding optimal codes.

We obtained 1316 mutually non-isomorphic residual designs of the 78 symmetric 2-(45, 12, 3) designs. These are designs with parameters 2-(33, 9, 3). In Table 5 we present an information about orders of their full automorphism groups. Incidence matrices of these designs span up to equivalence 627 ternary codes of length 33. In Table 6 we list the number of codes with given parameters and orders of the automorphism groups of these 627 codes.

5. Orbit matrices

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a symmetric (v, k, λ) design and $G \leq \mathrm{Aut}\,\mathcal{D}$. The group action of G produces the same number of point and block orbits (see [21, Theorem 3.3]). We denote that number by t, the point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, the block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. We shall denote the points of the orbit $\mathcal{P}_r$ by $r_0, \ldots, r_{\omega_{r-1}}$, (i.e. $\mathcal{P}_r = \{r_0, \ldots, r_{\omega_{r-1}}\}$). Further, we denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold (see [9]):

$$\sum_{r=1}^{t} \gamma_{ir} = k, \tag{4}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k-\lambda). \tag{5}$$

Table 6
Ternary codes of the residual designs.

Parameters	$\|\mathrm{Aut}(C_\Gamma)\|$	No	Parameters	$\|\mathrm{Aut}(C_\Gamma)\|$	No	Parameters	$\|\mathrm{Aut}(C_\Gamma)\|$	No	Parameters	$\|\mathrm{Aut}(C_\Gamma)\|$	No
[33, 14, 8]	2304	1	[33, 16, 7]	6	1		216	1		2	1
[33, 16, 3]	36	4	[33, 16, 8]	192	1		108	1	[33, 19, 3]	186 624	1
	18	6		48	1		54	1		139 968	1
	12	1		24	1		36	3		3 456	1
	6	1	[33, 17, 3]	2 592	1		18	1		864	2
	4	1		1 944	1		12	6		432	4
	2	1		54	1		6	19		324	1
	1	1		36	2		4	7		216	3
[33, 16, 4]	72	1		18	6		3	21		144	2
	16	1		12	9		2	53		108	2
	12	1		6	5		1	75		72	3
	8	2		4	1	[33, 18, 4]	24	1		36	1
	4	2		3	1		12	3		18	3
	2	8	[33, 17, 4]	72	1		8	2		12	22
	1	2		18	1		6	4		9	1
[33, 16, 6]	144	1		12	2		4	11		6	51
	72	1		8	1		2	32		4	14
	54	1		6	2		1	97		3	12
	36	1		2	1	[33, 18, 5]	2	3		2	24
	18	1	[33, 17, 6]	6	1	[33, 18, 6]	576	1		1	18
	16	1	[33, 18, 3]	559 872	1		72	1	[33, 19, 4]	8	1
	8	1		1 944	2		12	1		6	1
	6	1		864	1		6	2		4	4
	3	1		648	3		4	1		3	1
	2	1		432	2		3	1		2	13

Definition 1. A $(t \times t)$-matrix (γ_{ir}) with entries satisfying conditions (3) and (4) is called an orbit matrix for the parameters (v, k, λ) and orbit lengths distributions $(\omega_1, \ldots, \omega_t)$, $(\Omega_1, \ldots, \Omega_t)$.

Orbit matrices are often used in construction of designs with the presumed automorphism group. Construction of designs admitting an action of the presumed automorphism group consists of two basic steps (see [17]):

1. Construction of orbit matrices for the given automorphism group;
2. Construction of block designs for the orbit matrices obtained in this way. This step is often called an indexing of orbit matrices.

Remark 2. Note that given an orbit matrix M the rows and columns that correspond to non-fixed blocks and non-fixed points form a submatrix called the non-fixed part of the orbit matrix M.

6. Self-orthogonal codes from orbit matrices

The following theorem proved by Harada and Tonchev [16] gives a construction of self-orthogonal codes from orbit matrices of 2-designs.

Result 1 (*[16, Proposition 1]*)**.** *Let $\mathcal{D}$ be a 2-(v, k, λ) design admitting a fixed-point-free and fixed-block-free automorphism ϕ of order q, where q is prime. Further, let M be the orbit matrix induced by the action of the group $G = \langle \phi \rangle$ on the design $\mathcal{D}$. If p is a prime dividing r and λ then the orbit matrix M generates a self-orthogonal code of length $b|q$ over $\mathbb{F}_p$, where b is the number of blocks of $\mathcal{D}$.*

A similar result can be found in [28].

Result 2 (*[28, Theorem 1.113]*)**.** *If G is a cyclic group of a prime order p that does not fix any point or block and $p|(r - \lambda)$, then the rows of the orbit matrix M generate a self-orthogonal code over $\mathbb{F}_p$.*

Using the above results Harada and Tonchev in [16] classified all non-binary codes from some symmetric 2-(v, k, λ) designs admitting a fixed-point-free automorphism of prime order p. In particular, codes over $\mathbb{F}_5$ were obtained from orbit matrices of symmetric designs with parameters (45, 12, 3).

We give the following generalization of Result 2 for symmetric designs which will be applied in our study of the ternary codes obtained from the (45, 12, 3)-designs:

Theorem 4. *Let G be an automorphism group of a symmetric (v, k, λ) design $\mathcal{D}$. If G is a cyclic group of prime order p and $p|(r - \lambda)$, then the rows of the non-fixed part of the orbit matrix M generate a self-orthogonal code of length $\frac{v-f}{p}$ over $\mathbb{F}_p$, where f is the number of fixed points.*

Proof. Suppose that G acts on the design $\mathcal{D}$ with f fixed points. Let $t = f + \frac{v-f}{p}$ be the number of orbits of the group G. We can assume that $\omega_i = \Omega_i = 1$, for $i = 1, \dots, f$, and $\omega_i = \Omega_i = p$, for $i = f+1, \dots, f + \frac{v-f}{p}$. For $f + 1 \le i, j \le f + \frac{v-f}{p}$ we have

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \sum_{r=1}^{f} \frac{p}{1} \gamma_{ir}\gamma_{jr} + \sum_{r=f+1}^{f+\frac{v-f}{p}} \frac{p}{p} \gamma_{ir}\gamma_{jr} = \lambda p + \delta_{ij} \cdot (k - \lambda),$$

and therefore

$$\sum_{r=f+1}^{f+\frac{v-f}{p}} \gamma_{ir}\gamma_{jr} = p\left(\lambda - \sum_{r=1}^{f} \gamma_{ir}\gamma_{jr}\right) + \delta_{ij} \cdot (k - \lambda). \quad \square$$

Note that in Theorem 4 the matrix M does not have to be an orbit matrix induced by an automorphism group, it is sufficient that M is an orbit matrix in the sense of Definition 1. The same holds true for Results 1 and 2. Therefore we will examine the self-orthogonal codes from orbit matrices which satisfy conditions (3) and (4), regardless of whether or not they come from an action of a group on a design.

7. An automorphism of order 3 acting fixed-point-freely on (45, 12, 3) designs

The following result quoted from [21] gives bounds on the number $f(\sigma)$ of fixed points of an automorphism of a symmetric design.

Result 3. *Suppose that a nonidentity automorphism σ of a nontrivial symmetric (v, k, λ) design fixes $f(\sigma)$ points. Then $f(\sigma) \le v - 2n$ and $f(\sigma) \le \frac{\lambda v}{k - \sqrt{n}}$. Moreover, if equality holds in either inequality, σ must be an involution and every non-fixed block contains exactly λ fixed points.*

Further, $f(\sigma) \equiv v (\mathrm{mod}\ |\sigma|)$. Therefore if σ is an automorphism of a symmetric (45, 12, 3) design and $|\sigma| = 3$, then $f(\sigma) \in \{0, 3, 6, 9, 12\}$. Since for $f(\sigma) = 12$ there are no solutions for Eqs. (4) and (5), we cannot have an orbit matrix, and so we conclude that $f(\sigma) \in \{0, 3, 6, 9\}$.

Henceforth, as an application of Theorem 4 we construct and classify up to equivalence all ternary self-orthogonal codes of lengths 12, 13, 14, and 15 from orbit matrices of the symmetric 2-(45, 12, 3) designs admitting an automorphism of order 3.

We begin by examining the orbit matrices of symmetric (45, 12, 3) designs with a fixed-point-free automorphism of order 3. Solving Eqs. (4) and (5) we obtain up to isomorphism 293 orbit matrices for $\mathbb{Z}_3$ acting on symmetric (45, 12, 3) designs with no fixed points. These orbit matrices generate up to equivalence 47 ternary self-orthogonal codes of length 15, whose parameters and weight distributions are presented in Table 7. Notice that Table 7 is presented in two blocks and the codes are ordered according to the size of the automorphism group. A similar ordering is given for Tables 8 and 9. In Table 7 for each block the first column represents the numbering of the codes, the second column represents the code's parameters, the third column provides the order of the automorphism group of the code, and the remaining columns provide the number of codewords of a given weight. The codes with parameters $[15, 4, 9]_3$ and $[15, 7, 6]$ are optimal, and those with parameters $[15, 6, 6]_3$ are a distance 1 less than the optimal; see [14]. We have the following result.

Proposition 5. *The ternary self-orthogonal codes of length* 15 *obtained from the orbit matrices of symmetric* (45, 12, 3) *designs with fixed-point-free automorphism group of order* 3 *are divided into forty-seven equivalence classes. Two of these codes are optimal.*

8. An automorphism of order 3 acting on (45, 12, 3) designs with three fixed points

Up to isomorphism there are 245 orbit matrices for $\mathbb{Z}_3$ acting on symmetric (45, 12, 3) designs with three fixed points. Non-fixed parts of these orbit matrices span up to equivalence 17 ternary self-orthogonal codes of length 14. Those 17 codes are presented in Table 8. The codes with parameters $[14, 6, 6]_3$ are optimal, and those with parameters $[14, 5, 6]_3$ are a distance 1 less than the optimal; see [14].

Proposition 6. *The ternary self-orthogonal codes of length* 14 *obtained from the orbit matrices of symmetric* (45, 12, 3) *designs having three fixed points are divided into seventeen equivalence classes. Three of these codes are optimal.*

Table 7
Weight distributions of the ternary codes of length 15.

No.	Parameters	$\lvert\mathrm{Aut}(C)\rvert$	0	3	6	9	12	15	No.	Parameters	$\lvert\mathrm{Aut}(C)\rvert$	0	3	6	9	12	15
1	$[15, 3, 6]_3$	933 120	1		6	8	6	6	25	$[15, 5, 6]_3$	18	1		26	110	102	4
2	$[15, 4, 6]_3$	933 120	1		20	20	30	10	26	$[15, 6, 3]_3$	18	1	2	42	392	292	
3	$[15, 7, 3]_3$	23 328	1	10	256	944	944	32	27	$[15, 6, 6]_3$	16	1		46	392	288	2
4	$[15, 6, 3]_3$	2 916	1	8	78	302	340		28	$[15, 6, 3]_3$	12	1	2	56	368	298	4
5	$[15, 6, 3]_3$	2 592	1	4	28	422	266	8	29	$[15, 6, 6]_3$	12	1		56	380	282	10
6	$[15, 7, 3]_3$	1 296	1	6	176	1100	888	16	30	$[15, 6, 6]_3$	12	1		54	386	276	12
7	$[15, 7, 3]_3$	1 296	1	4	136	1178	860	8	31	$[15, 6, 6]_3$	8	1		40	410	270	8
8	$[15, 7, 3]_3$	720	1	2	180	1112	868	24	32	$[15, 6, 6]_3$	8	1		58	374	288	8
9	$[15, 6, 3]_3$	648	1	2	84	320	310	12	33	$[15, 5, 6]_3$	6	1		14	134	90	4
10	$[15, 5, 3]_3$	648	1	2	24	104	112		34	$[15, 6, 6]_3$	6	1		48	386	294	
11	$[15, 7, 3]_3$	432	1	2	138	1184	850	12	35	$[15, 6, 3]_3$	6	1	2	42	392	292	
12	$[15, 6, 3]_3$	216	1	4	82	314	320	8	36	$[15, 6, 3]_3$	6	1	2	38	404	280	4
13	$[15, 7, 6]_3$	168	1		140	1190	840	16	37	$[15, 5, 6]_3$	6	1		12	134	96	
14	$[15, 5, 6]_3$	108	1		24	116	96	6	38	$[15, 5, 6]_3$	4	1		8	146	84	4
15	$[15, 6, 3]_3$	108	1	2	42	392	292		39	$[15, 5, 6]_3$	4	1		12	134	96	
16	$[15, 4, 6]_3$	108	1		6	38	36		40	$[15, 6, 6]_3$	4	1		44	398	282	4
17	$[15, 5, 6]_3$	96	1		30	104	102	6	41	$[15, 6, 6]_3$	2	1		44	398	282	4
18	$[15, 6, 3]_3$	72	1	2	48	392	274	12	42	$[15, 5, 6]_3$	2	1		12	134	96	
19	$[15, 5, 3]_3$	54	1	2	24	104	112		43	$[15, 5, 6]_3$	2	1		16	128	96	2
20	$[15, 5, 3]_3$	54	1	2	6	140	94		44	$[15, 6, 6]_3$	2	1		42	404	276	6
21	$[15, 6, 6]_3$	48	1		62	362	300	4	45	$[15, 5, 6]_3$	1	1		10	140	90	2
22	$[15, 6, 6]_3$	48	1		60	368	294	6	46	$[15, 5, 6]_3$	1	1		12	134	96	
23	$[15, 6, 3]_3$	24	1	2	52	380	286	8	47	$[15, 5, 6]_3$	1	1		10	140	90	2
24	$[15, 4, 9]_3$	18	1			50	30										

Table 8
Weight distributions of the ternary codes of length 14.

No.	Parameters	$\lvert\mathrm{Aut}(C)\rvert$	0	3	6	9	12
1	$[14, 4, 6]_3$	4320	1		24	20	36
2	$[14, 5, 3]_3$	432	1	2	24	158	58
3	$[14, 5, 6]_3$	216	1		42	128	72
4	$[14, 6, 3]_3$	216	1	2	96	446	184
5	$[14, 6, 3]_3$	216	1	2	78	482	166
6	$[14, 5, 6]_3$	216	1		48	116	78
7	$[14, 6, 6]_3$	72	1		102	440	186
8	$[14, 4, 6]_3$	36	1		6	56	18
9	$[14, 6, 6]_3$	24	1		84	476	168
10	$[14, 5, 6]_3$	12	1		30	152	60
11	$[14, 6, 3]_3$	12	1	6	84	458	180
12	$[14, 5, 6]_3$	6	1		24	164	54
13	$[14, 6, 6]_3$	6	1		84	476	168
14	$[14, 5, 6]_3$	4	1		24	164	54
15	$[14, 5, 3]_3$	4	1	2	24	158	58
16	$[14, 6, 3]_3$	4	1	4	72	488	164
17	$[14, 5, 6]_3$	2	1		24	164	54

Table 9
Weight distributions of the ternary codes of length 13.

No.	Parameters	$\lvert\mathrm{Aut}(C)\rvert$	0	3	6	9	12
1	$[13, 4, 6]_3$	1728	1		24	38	18
2	$[13, 5, 3]_3$	324	1	2	42	176	22
3	$[13, 6, 3]_3$	216	1	2	150	500	76
4	$[13, 5, 6]_3$	144	1		54	158	30
5	$[13, 6, 3]_3$	24	1	6	138	512	72
6	$[13, 6, 6]_3$	24	1		156	494	78
7	$[13, 5, 6]_3$	6	1		48	170	24

9. An automorphism of order 3 acting on $(45, 12, 3)$ designs with six fixed points

There are 49 mutually non-isomorphic solutions of Eqs. (4) and (5) for $\mathbb{Z}_3$ acting on symmetric $(45, 12, 3)$ designs with six fixed points. These 49 orbit matrices generate up to equivalence 7 ternary self-orthogonal codes of length 13. The codes are listed in Table 9. Note that the codes (#1, 4, 6 and 7) have minimum weight 6, and thus are optimal, see [14].

Proposition 7. *The ternary self-orthogonal codes of length* 13 *obtained from the orbit matrices of symmetric* (45, 12, 3) *designs having six fixed points are divided into seven equivalence classes. Four of these codes are optimal.*

10. An automorphism of order 3 acting on (45, 12, 3) designs with nine fixed points

Up to isomorphism there are four orbit matrices for an automorphism group of order 3 acting on a (45, 12, 3) design with nine fixed points. The ternary codes generated by these four orbit matrices are mutually equivalent, and a representative of this class is the self-orthogonal three-weight $[12, 3, 6]_3$ code. This is a quasi-cyclic code with distance 2 less than the optimal, and automorphism group of order 31 104. The weight enumerator is

$$W(x) = 1 + 12x^6 + 8x^9 + 6x^{12}.$$

The dual code is a $[12, 9, 2]_3$ code with weight enumerator

$$W(x) = 1 + 24x^2 + 8x^3 + 378x^4 + 792x^5 + 2508x^6 + 3456x^7 + 4662x^8 + 4352x^9 + 2484x^{10} + 840x^{11} + 178x^{12}.$$

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Further reading

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